

BOXICITY AND TOPOLOGICAL INVARIANTS

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ABSTRACT. The boxicity of a graph $G = (V, E)$ is the smallest integer k for which there exist k interval graphs $G_i = (V, E_i)$, $1 \leq i \leq k$, such that $E = E_1 \cap \dots \cap E_k$. In the first part of this note, we prove that every graph on m edges has boxicity $O(\sqrt{m \log m})$, which is asymptotically best possible. We use this result to study the connection between the boxicity of graphs and their Colin de Verdière invariant, which share many similarities. Known results concerning the two parameters suggest that for any graph G , the boxicity of G is at most the Colin de Verdière invariant of G , denoted by $\mu(G)$. We observe that every graph G has boxicity $O(\mu(G)^4 (\log \mu(G))^2)$, while there are graphs G with boxicity $\Omega(\mu(G) \sqrt{\log \mu(G)})$. In the second part of this note, we focus on graphs embeddable on a surface of Euler genus g . We prove that these graphs have boxicity $O(\sqrt{g} \log g)$, while some of these graphs have boxicity $\Omega(\sqrt{g} \log g)$. This improves the previously best known upper and lower bounds. These results directly imply a nearly optimal bound on the dimension of the adjacency poset of graphs on surfaces.

1. INTRODUCTION

Given a collection \mathcal{C} of subsets of a set Ω , the *intersection graph* of \mathcal{C} is defined as the graph with vertex set \mathcal{C} , in which two elements of \mathcal{C} are adjacent if and only if their intersection is non empty. A *d-box* is the Cartesian product $[x_1, y_1] \times \dots \times [x_d, y_d]$ of d closed intervals of the real line. The *boxicity* $\text{box}(G)$ of a graph G , introduced by Roberts [14] in 1969, is the smallest integer $d \geq 1$ such that G is the intersection graph of a collection of d -boxes. The *intersection* $G_1 \cap \dots \cap G_k$ of k graphs G_1, \dots, G_k defined on the same vertex set V , is the graph $(V, E_1 \cap \dots \cap E_k)$, where E_i ($1 \leq i \leq k$) denotes the edge set of G_i . Observe that the boxicity of a graph G can equivalently be defined as the smallest k such that G is the intersection of k interval graphs.

In the first part of this note, we prove that every graph on m edges has boxicity $O(\sqrt{m \log m})$, and that there are examples showing that this bound is asymptotically best possible.

A minor-monotone graph invariant, usually denoted by $\mu(\cdot)$, was introduced by Colin de Verdière in 1990 [4]. It relates to the maximal multiplicity of the second largest eigenvalue of the adjacency matrix of a graph, in which the diagonal entries can take any value and the entries corresponding to edges can take any positive values (a technical assumption, called the Strong Arnold Property, has to be added to avoid degenerate cases, but we omit the details as they are not necessary in our discussion).

It was proved by Colin de Verdière that $\mu(G) \leq 1$ if and only if G is a linear forest, $\mu(G) \leq 2$ if and only if G is an outerplanar graph, and $\mu(G) \leq 3$ if and only if G is a planar graph. Scheinerman proved in 1984 that outerplanar graphs have boxicity at most two [15] and Thomassen proved in 1986 that planar graphs have boxicity at most three [17]. Since a linear forest is an interval graph, these results prove that for any planar graph G , $\text{box}(G) \leq \mu(G)$.

These two graph invariants share several other similarities: every graph G of treewidth at most k has $\text{box}(G) \leq k + 1$ [3] and $\mu(G) \leq k + 1$ [9]. For any vertex v of G , $\text{box}(G - v) \leq \text{box}(G) + 1$

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and if $G - v$ contains an edge, $\mu(G - v) \leq \mu(G) + 1$. Both parameters are bounded for graphs G with crossing number at most k : $\text{box}(G) = O(k^{1/4}(\log k)^{3/4})$ [2] and $\mu(G) \leq k + 3$ [4]. It is known that every graph on n vertices has boxicity at most $n/2$, and equality holds only for complements of perfect matchings [14]. These graphs have Colin de Verdière invariant at least $n - 3$ [11]. On the other hand every graph on n vertices has Colin de Verdière invariant at most $n - 1$, and equality holds only for cliques (which have boxicity 1).

It is interesting to note that in each of the results above, the known upper bound on the boxicity is better than the known upper bound on the Colin de Verdière invariant. This suggests that for any graph G , $\text{box}(G) \leq \mu(G)$.

The following slightly weaker relationship between the boxicity and the Colin de Verdière invariant is a direct consequence of the fact that any graph G excludes the clique on $\mu(G) + 2$ vertices as a minor, and graphs with no K_t -minor have boxicity $O(t^4(\log t)^2)$ [6].

Proposition 1. *There is a constant c_0 such that for any graph G , $\text{box}(G) \leq c_0 \mu(G)^4 (\log \mu(G))^2$.*

It follows that the boxicity is bounded by a polynomial function of the Colin de Verdière invariant.

Pendavingh [13] proved that for any graph G with m edges, $\mu(G) \leq \sqrt{2m}$. Interestingly, there did not exist any corresponding result for the boxicity and it was suggested by András Sebő that graphs G with large boxicity (as a function of their number of edges) might satisfy $\text{box}(G) > \mu(G)$. As we observe in the next section, there are graphs on m edges, with boxicity $\Omega(\sqrt{m \log m})$. It follows that there are graphs G with boxicity $\Omega(\mu(G) \sqrt{\log \mu(G)})$. These graphs show that the boxicity is not even bounded by a linear function of the Colin de Verdière invariant.

In the second part of this paper, we show that every graph embeddable on a surface of Euler genus g has boxicity $O(\sqrt{g \log g})$, while there are graphs embeddable on a surface of Euler genus g with boxicity $\Omega(\sqrt{g \log g})$. This improves the upper bound $O(g)$ and the lower bound $\Omega(\sqrt{g})$ given in [6]. (Incidentally, graphs embeddable on a surface of Euler genus g have Colin de Verdière invariant $O(g)$ and it is conjectured that the right bound should be $O(\sqrt{g})$ [4, 16].)

Our upper bound on the boxicity of graphs on surfaces has a direct corollary on the dimension of the adjacency poset of graphs on surfaces, introduced by Felsner and Trotter [8], and investigated in [7] and [6].

2. BOXICITY AND THE NUMBER OF EDGES

We will use the following two lemmas of Adiga, Chandran, and Mathew [2]. A graph G is *k-degenerate* if every subgraph of G contains a vertex of degree at most k . In what follows, the logarithm is taken to be the natural logarithm (and its base is denoted by e).

Lemma 2. [2] *Any k -degenerate graph on $n \geq 2$ vertices has boxicity at most $(k + 2) \lceil 2e \log n \rceil$.*

Lemma 3. [2] *Let G be a graph, and let S be a set of vertices of G . Let H be the graph obtained from G by removing all edges between pairs of vertices of S . Then $\text{box}(G) \leq 2 \text{box}(H) + \text{box}(G[S])$, where $G[S]$ stands for the subgraph of G induced by S .*

We now prove that every graph on m vertices has boxicity $O(\sqrt{m \log m})$. We make no real effort to optimize the constants, and instead focus on simplifying the computation as much as possible.

Theorem 4. *For every graph G on $n \geq 2$ vertices and m edges, $\text{box}(G) \leq (15e + 1) \sqrt{m \log n}$.*

Proof. Let $G = (V, E)$ be a graph on $n \geq 2$ vertices and m edges. Since the boxicity of a graph is the maximum boxicity of its connected components, we can assume that G is connected (in particular, $m \geq n - 1 \geq \log n$). Let S be a set of vertices of G obtained as follows: start with

$S = V$ and as long as S contains a vertex v with at most $\sqrt{m/\log n}$ neighbors in S , remove v from S . Let H be the graph obtained from G by removing all edges between pairs of vertices of S .

The order in which the vertices were removed from S shows that the graph H is $\sqrt{m/\log n}$ -degenerate. By Lemma 2, for $k \geq 1$, every k -degenerate graph on $n \geq 2$ vertices has boxicity at most $(k+2)\lceil 2e \log n \rceil \leq \frac{15e}{2}k \log n$. It follows that H has boxicity at most $\frac{15e}{2}\sqrt{m \log n}$.

By definition of S , every vertex of S has degree more than $\sqrt{m/\log n}$ in $G[S]$, so $G[S]$ has at least $\frac{|S|}{2}\sqrt{m/\log n}$ edge. It follows that $|S| \leq 2\sqrt{m \log n}$. Since any graph on N vertices has boxicity at most $N/2$ [14], $\text{box}(G[S]) \leq \sqrt{m \log n}$.

By Lemma 3, $\text{box}(G) \leq 2\text{box}(H) + \text{box}(G[S])$. It follows that G has boxicity at most $15e\sqrt{m \log n} + \sqrt{m \log n} = (15e+1)\sqrt{m \log n}$, as desired. \square

Remark 5. As proved in [2], Lemmas 2 and 3 also hold if the boxicity is replaced by the *cubicity* (the smallest k such that G is the intersection of k unit-interval graphs), so the proof can easily be adapted to show that any graph G with m edges has cubicity $O(\sqrt{m \log m})$.

We now observe that Theorem 4 is asymptotically best possible. Let G_n be a bipartite graph with n vertices in each partite set, and such that every edge between the two partite sets is selected uniformly at random with probability $p = 1/\log n$. Using Chernoff bound, it is easy to deduce that asymptotically almost surely (i.e., with probability tending to 1 as n tends to infinity) G_n has at most $2n^2/\log n$ edges. Using a nice connection between the dimension of a poset and the boxicity of its comparability graph [1], Adiga, Bhowmick and Chandran deduced from a result of Erdős, Kierstead and Trotter [5] that there is a constant $c_1 > 0$ such that asymptotically almost surely, $\text{box}(G_n) \geq c_1 n$ (see also [2]). It follows that asymptotically almost surely, $\text{box}(G_n) \geq c_1 \sqrt{|E(G_n)| \log n/2}$, which shows the (asymptotic) optimality of Theorem 4.

Recall that by [13], $\mu(G_n) \leq \sqrt{2|E(G_n)|}$. This implies the following counterpart of Proposition 1.

Proposition 6. *For some constant $c'_1 > 0$, there are infinitely many graphs G with $\text{box}(G) \geq c'_1 \mu(G) \sqrt{\log \mu(G)}$.*

Remark 7. As mentioned earlier, it was proved in [6] that graphs with no K_t -minor have boxicity $O(t^4(\log t)^2)$. Since the size of a largest clique minor in G_n is at most $\mu(G_n) + 1$, the discussion above implies the existence of graphs with no K_t -minor and with boxicity $\Omega(t\sqrt{\log t})$.

3. BOXICITY AND ACYCLIC COLORING OF GRAPHS ON SURFACES

In this paper, a *surface* is a non-null compact connected 2-manifold without boundary. We refer the reader to the book by Mohar and Thomassen [12] for background on graphs on surfaces.

A surface can be orientable or non-orientable. The *orientable surface* \mathbb{S}_h of *genus* h is obtained by adding $h \geq 0$ *handles* to the sphere; while the *non-orientable surface* \mathbb{N}_k of *genus* k is formed by adding $k \geq 1$ *cross-caps* to the sphere. The *Euler genus* of a surface Σ is defined as twice its genus if Σ is orientable, and as its non-orientable genus otherwise.

The following is a direct consequence of [12, Proposition 4.4.4].

Lemma 8. [12] *If a graph G embedded in a surface of Euler genus g contains the complete bipartite graph $K_{3,k}$ as a subgraph, then $k \leq 2g + 2$.*

A coloring of the vertices of a graph is said to be *acyclic* if it is proper (any two adjacent vertices have different colors) and any two color classes induce a forest. The following result of [6] relates acyclic coloring and boxicity.

Lemma 9. [6] *If G has an acyclic coloring with $k \geq 2$ colors, then $\text{box}(G) \leq k(k-1)$.*

We will also use the following recent result of Kawarabayashi and Thomassen [10] (note that the constant 1000 can easily be improved).

Theorem 10. [10] *Any graph G embedded in a surface of Euler genus g contains a set A of at most $1000g$ vertices such that $G - A$ has an acyclic coloring with 7 colors.*

Note that combining Lemma 9 and Theorem 10, it is not difficult to derive that graphs embedded in a surface of Euler genus g have boxicity at most $500g + 42$ (a linear bound with better constants was given in [6], using completely different arguments). We will now prove instead that their boxicity is $O(\sqrt{g} \log g)$.

Given a graph G and a subset A of vertices of G , the A -neighborhood of a vertex v of $G - A$ is the set of neighbors of v in A . We are now ready to prove the main result of this section.

Theorem 11. *There is a constant c_2 such that any graph embedded in a surface of Euler genus $g \geq 2$ has boxicity at most $c_2 \sqrt{g} \log g$.*

Proof. Let $G = (V, E)$ be a graph embedded in a surface of Euler genus g . By Theorem 10, G contains a set A of at most $1000g$ vertices such that $G - A$ has an acyclic coloring with at most 7 colors. In particular, Lemma 9 implies that $G - A$ has boxicity at most 42.

Let H be the graph obtained from G by deleting all edges between pairs of vertices of $V - A$, and then identifying any two vertices of $V - A$ having the same A -neighborhood. Since any two vertices of $H - A$ have distinct A -neighborhoods, $H - A$ contains at most $1 + |A| + \binom{|A|}{2}$ vertices having at most two neighbors in A . Let x, y, z be three vertices of A . By Lemma 8, at most $2g + 2$ vertices of $H - A$ are adjacent to each of x, y, z . It follows that $H - A$ contains at most $1 + |A| + \binom{|A|}{2} + (2g + 2) \binom{|A|}{3} \leq 1 + |A| + \frac{1}{2}|A|^2 + \frac{1}{6}|A|^3(2g + 2) \leq 10^9 g^4$ vertices. It was proved by Heawood (see [12, Theorem 8.3.1]) that every graph embeddable on a surface of Euler genus g is $\frac{1}{2}(5 + \sqrt{1 + 24g})$ -degenerate. Consequently, H (as a subgraph of G) is $\frac{1}{2}(5 + \sqrt{1 + 24g})$ -degenerate and by Lemma 2, it has boxicity at most

$$\left(\frac{1}{2}(5 + \sqrt{1 + 24g}) + 2 \right) \lceil 2e \log(10^9 g^4 + 10^3 g) \rceil \leq c_3 \sqrt{g} \log g,$$

for some constant c_3 .

Let H_1 be the graph obtained from H by adding all edges between pairs of vertices of $H - A$. Since $H_1 - A$ is a complete graph, $\text{box}(H_1 - A) = 1$. By Lemma 3, $\text{box}(H_1) \leq 2 \text{box}(H) + \text{box}(H_1 - A) \leq 2c_3 \sqrt{g} \log g + 1$.

Let G_1 be the graph obtained from G by adding all edges between pairs of vertices of $V - A$. It is clear that $\text{box}(G_1) \leq \text{box}(H_1)$, since any two vertices of $G_1 - A$ having the same A -neighborhood are adjacent and have the same neighborhood in G_1 , so they can be mapped to the same d -box in a representation of G_1 as an intersection of d -boxes. Hence, $\text{box}(G_1) \leq 2c_3 \sqrt{g} \log g + 1$.

Let G_2 be the graph obtained from G by adding all edges between a vertex of A and a vertex of V (i.e. G_2 is obtained from $G[V - A]$ by adding $|A|$ universal vertices). Clearly, $\text{box}(G_2) \leq \text{box}(G[V - A]) \leq 42$. The graphs G_1 and G_2 are supergraphs of G , and any non-edge of G appears in G_1 or G_2 , so $\text{box}(G) \leq \text{box}(G_1) + \text{box}(G_2) \leq 2c_3 \sqrt{g} \log g + 43$. It follows that there is a constant c_2 , such that $\text{box}(G) \leq c_2 \sqrt{g} \log g$, as desired. \square

Recall the probabilistic construction mentioned at the end of Section 2: there is a sequence of random graphs G_n on $2n$ vertices, such that asymptotically almost surely G_n has at most

$2n^2/\log n$ edges and boxicity at least $c_1 n$, for some universal constant $c_1 > 0$. It directly follows from Euler Formula that the Euler genus of a graph is at most its number of edges plus 2, so asymptotically almost surely, G_n has Euler genus at most $2n^2/\log n + 2$. Hence, asymptotically almost surely, $\text{box}(G_n) \geq c_1 n \geq c_1 \sqrt{g \log n/2} \geq \frac{c_1}{2} \sqrt{g \log g}$, where g stands for the Euler genus of G_n . Consequently, the bound of Theorem 11 is optimal up to a factor of $\sqrt{\log g}$.

The *adjacency poset* of a graph $G = (V, E)$, introduced by Felsner and Trotter [8], is the poset (W, \leq) with $W = V \cup V'$, where V' is a disjoint copy of V , and such that $u \leq v$ if and only if $u = v$, or $u \in V$ and $v \in V'$ and u, v correspond to two distinct vertices of G which are adjacent in G . The *dimension* of a poset \mathcal{P} is the minimum number of linear orders whose intersection is exactly \mathcal{P} . It was proved in [6] that for any graph G , the dimension of the adjacency poset of G is at most $2\text{box}(G) + \chi(G) + 4$, where $\chi(G)$ is the chromatic number of G . Since graphs embedded on a surface of Euler genus g have chromatic number $O(\sqrt{g})$, we obtain the following corollary of Theorem 11, which improves the linear bound obtained in [6] and is best possible up to a logarithmic factor.

Corollary 12. *There is a constant c_3 such that for any graph G embedded in a surface of Euler genus $g \geq 2$, the dimension of the adjacency poset of G is at most $c_3 \sqrt{g} \log g$.*

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